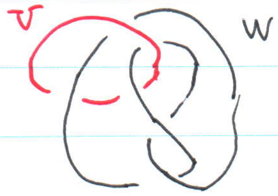
Lauda $sl(3)$ link homology

Back ground

$\mathfrak{g}$: simple Lie alg.

each coloring of L by irreducible representation of $\mathfrak{g}$ gives an invariant



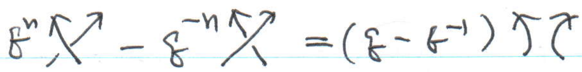
V, W : irrep. of $\mathfrak{g}$

Reshetikhin – Turaev using $U_q(\mathfrak{g})$

sl_2, std 2-dim. rep. $\longrightarrow$ Jones poly.

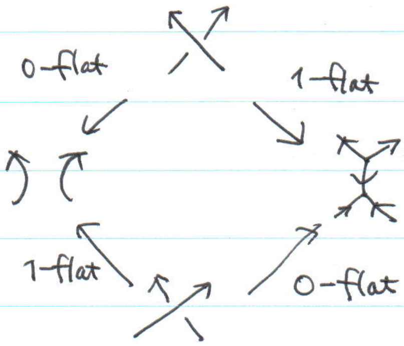
sl_n, std n-dim. rep. $\longrightarrow$ HOMFLY-PT pol

HOMFLY-PT

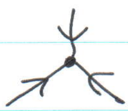


A 2-variable pol. $a = q^n$, $t = q - q^{-1}$

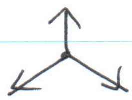
Kuperberg has a nice graphical calculus for sl_3



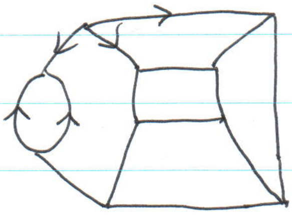
resolving all crossing gives planar graph



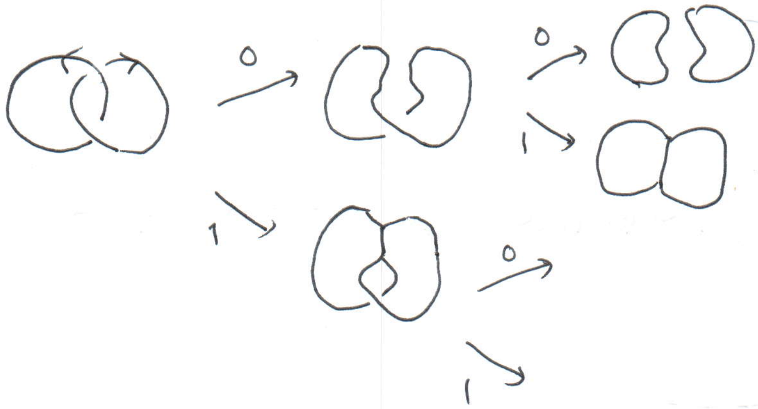
or

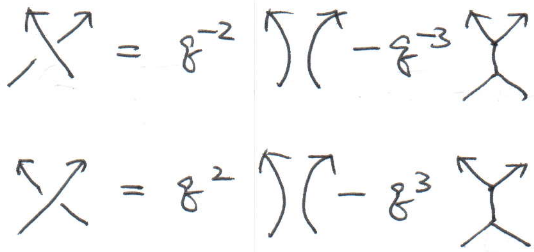


web :



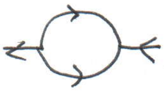
$\leftarrow$ a resolution of link





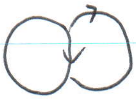
Kuperberg bracket $\langle \Gamma \rangle \in \mathbb{Z}[q, q^{-1}]$ Γ = web

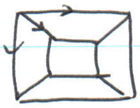
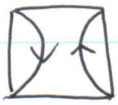
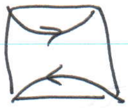
can remove loops  $= [3] = q^2 + 1 + q^{-2}$

 $= [2]$  $= (q + q^{-1})$ 

For web Γ $\langle \Gamma \rangle \in \underset{\sim}{\mathbb{Z}^{+}}[q, q^{-1}]$.

Hint : categorification of $\langle \Gamma \rangle$
will be a graded abelian group

$\langle$  $\rangle = [2]\langle$  $\rangle = [2][3]$

$\langle$  $\rangle =$  $+$ 

$= 2[2][2][3]$

categorification

$\langle \Gamma \rangle \longrightarrow H(\Gamma) \leftarrow$ graded abelian group

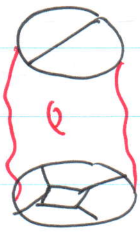
s.t. $\langle \Gamma \rangle = \mathrm{qrank}\, H(\Gamma)$

$\Gamma = \phi \rightsquigarrow H(\phi) = \mathbb{Z}$

$\Gamma =$  $\quad H(\Gamma)$

we want web cobordisms

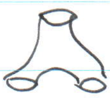
Γ_2

S  to induce maps

Γ_1

$$H(\Gamma_1) \xrightarrow[H(S)]{} H(\Gamma_2)$$

$\therefore$ $H(\Gamma)$: comm. Frob. ring $\quad \langle$  $\rangle = q^2 + 1 + q^{-2}$



$\mathbb{Z}$ deg 2

$\mathbb{Z}$ deg 0

$\mathbb{Z}$ deg -2

$A = H^*(\mathbb{C}P^2; \mathbb{Z})$ $\quad x^2 \quad 4$

$= \mathbb{Z}[x]/x^3$ $\quad x \quad 2$

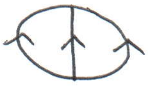
$\quad 1 \quad$ deg 0

<u>trace</u> $\varepsilon(1)=0\ ,\ \varepsilon(X)=0\ ,\ \varepsilon(X^2)=-1$

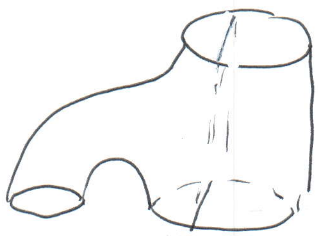
opposite orientation

This also tells us how to evaluate

 $\mapsto A^{\otimes k}$ $\quad k=\#$ of circles

$\Gamma=$  $\quad \langle\Gamma\rangle=(q+q^{-1})(q^2+1+q^{-2})=[2][3]$

We also know that $H(\Gamma)$ should be an A-module in 3 compatible ways



Recall Fl_3 = full flag mfd in $\mathbb{C}^3$

$=\{0\subset L_1\subset L_2\subset\mathbb{C}^3 \quad \dim L_i=i\}$

$$\begin{array}{ccc}\mathbb{P}^1 & \to & Fl_3\\ & & \downarrow\\ & & \mathbb{CP}^2\end{array}$$

$$\mathrm{rk}\ H^*(Fl_3;\mathbb{Z})=(1+q^2+q^4)(1+q^2)=[2][3]q^3$$

$Fl_3\cong\{(w_1,w_2,w_3)\mid w_i:\text{line in }\mathbb{C}^3 \quad w_i\perp w_j\}$

$\Rightarrow\ Fl_3\subset\mathbb{CP}^2\times\mathbb{CP}^2\times\mathbb{CP}^2$

$$\therefore\ \underset{\substack{\parallel\wr\\ A^{\otimes 3}}}{H^*(\mathbb{CP}^2\times\mathbb{CP}^2\times\mathbb{CP}^2)}\longrightarrow H^*(Fl_3) \quad \cdots \text{action}$$

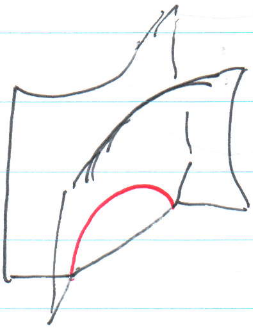
$$H(\text{theta web}) = \underbrace{H^*(Fl_3, \mathbb{Z})}_{\substack{\parallel \\ B = \mathbb{Z}[X_1, X_2, X_3]/\text{symm. poly}}}\{3\}$$

$X_1 + X_2 + X_3 = 0$
$X_1X_2 + X_1X_3 + X_2X_3 = 0$
$X_1X_2X_3 = 0$

$\rightsquigarrow$ H(web) determined

$$\text{crossing} = q^{-2}\,\uparrow\uparrow - q^{-3}\,\text{web}$$

$$H(\text{crossing}) = 0 \to H(\uparrow\uparrow)\{-2\} \to H(\text{web})\{-3\} \to 0$$



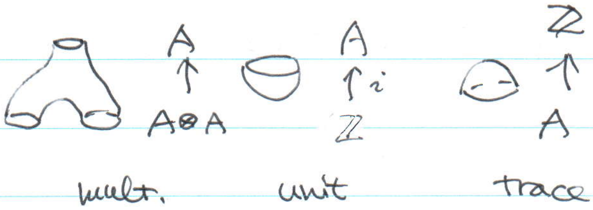
"foams" only [singular seam] sing.

E saddle の代わりにする．
sl_2 の

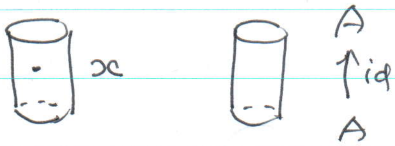
$H(\phi) = \mathbb{Z}$
$H(\bigcirc) = A$
$H(\ominus) = B$

These choices produce relations on foams!

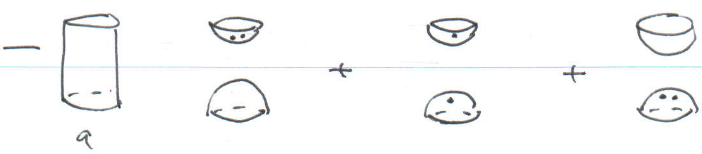
On nonsingular cobordisms



we can represent mult. by x as
gen. of A



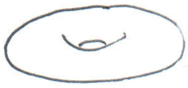
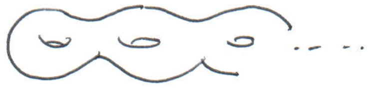
$a \in A$ $\quad -a = x^2 tr(a) + x\,tr(xa) + tr(x^2 a)$



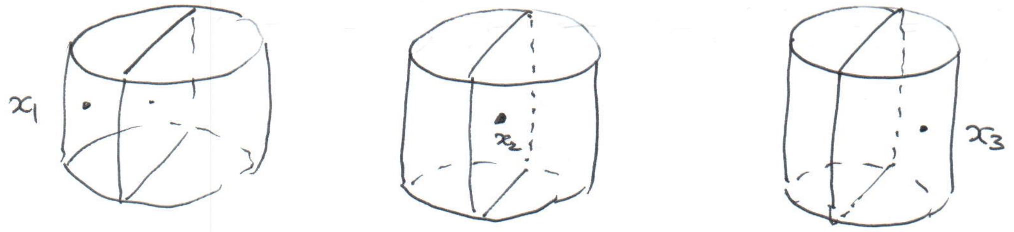
neck cut rel.

 $= 0$  $= 0$  $= -1$

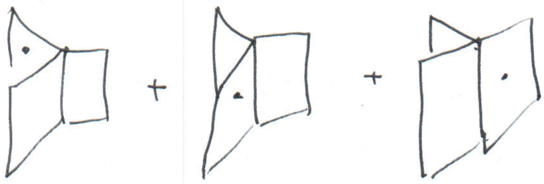
$tr(1)$

 $= 3 = \dim A$  $= 0$

We will represent multiplication by generators x_1, x_2, x_3



Looking locally, symmetric polynomials imply

 $= 0$

etc

- Using these rules we can assign to each closed form U an integer $\mathcal{F}(U)$

- To a web Γ we assign abelian group generated by symbols $\mathcal{F}(U)$ for all forms from the empty ~~set~~ into Γ

with relation $\sum a_i \mathcal{F}(U) = 0$ $\quad a_i \in \mathbb{Z}$
iff for any foam V from $\Gamma \to \phi$
$\sum a_i \mathcal{F}(VU) = 0$

<u>Thm</u>. $\mathcal{F}(\Gamma)$ is a graded abelian group

$$\operatorname{grank} \mathcal{F}(\Gamma) = \langle \Gamma \rangle$$